Problem Set 3 due March 11, at 10 PM, on Gradescope

Please list all of your sources: collaborators, written materials (other than our textbook and lecture notes) and online materials (other than Gilbert Strang's videos on OCW).

Give complete solutions, providing justifications for every step of the argument. Points will be deducted for insufficient explanation or answers that come out of the blue.

Problem 1:

(1) Given numbers a and b, for which number c does the system:

$$\begin{bmatrix} 1 & 2 \\ 1 & 1 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

have a solution v_1, v_2 (you must express c in terms of a and b). *(10 points)*

(2) Draw the set of vectors $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$ satisfing the conditions in part (1) on a picture of $\mathbb{R}^3$. *(5 points)*

(3) Construct a 3×3 matrix whose column space is spanned by $\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$. *(5 points)*

Solution: (1) We may row reduce the augmented matrix

$$\begin{bmatrix} 1 & 2 & a \\ 1 & 1 & b \\ 2 & 3 & c \end{bmatrix} \xrightarrow{r_2 - r_1} \begin{bmatrix} 1 & 2 & a \\ 0 & -1 & b-a \\ 2 & 3 & c \end{bmatrix} \xrightarrow{r_3 - 2r_1} \begin{bmatrix} 1 & 2 & a \\ 0 & -1 & b-a \\ 0 & -1 & c-2a \end{bmatrix} \xrightarrow{r_3 - r_2} \begin{bmatrix} 1 & 2 & a \\ 0 & -1 & b-a \\ 0 & 0 & c-b-a \end{bmatrix}$$

From this we learn that the original system of linear equations is equivalent to the system:

$$\begin{bmatrix} 1 & 2 \\ 0 & -1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} a \\ b-a \\ c-b-a \end{bmatrix} \quad \Leftrightarrow \quad \begin{cases} v_1 + 2v_2 = a \\ -v_2 = b - a \\ 0 = c - b - a \end{cases}$$

This system has a solution if and only if $c = a + b$.

Grading Rubric:

- Correct and complete argument (even if different from the one above) *(10 points)*
- Performed row reduction correctly, but did not reach conclusion *(6 points)*
- Performed row reduction with mistakes, and did not reach conclusion *(3 points)*

- Missing or wrong answer *(0 points)*

(2) Students should draw a plane in abc space with clearly labeled axes. It must be unequivocal which plane they are referring to, either by marking it as perpendicular to the vector $(1, 1, -1)$, or by marking its intersections with the coordinate planes.

Grading Rubric:

- Correct picture *(5 points)*
- Minor computational errors *(3 points)*
- Missing or wrong picture *(0 points)*

(3) An example is:

$$\begin{bmatrix} 3 & 1 & 2 \\ 2 & 1 & 1 \\ 5 & 2 & 3 \end{bmatrix}$$

As the first column is a linear combination (specifically the sum) of the other two columns, this 3×3 matrix has the column space we're looking for. Many other examples are possible.

Grading Rubric:

- Correct matrix with acknowledgement of the fact that the extra column is a linear combination of the other two columns *(5 points)*
- Correct matrix with no explanation *(3 points)*
- Incorrect matrix *(0 points)*

Problem 2:

Consider the following planes in 3–dimensional space:

- the xz coordinate plane
- the plane cut out by the equation $2x + 3y = 5z$

(1) Write down a 3×3 matrix A whose nullspace is the <u>intersection</u> of the two planes above. *(10 points)*

(2) Give examples of vectors $\boldsymbol{b} \in \mathbb{R}^3$ for which the system of equations $A\boldsymbol{v} = \boldsymbol{b}$ has:

- no solutions *(5 points)*

- infinitely many solutions *(5 points)*

Solution: (1) The xz coordinate plane is cut out by the equation $y = 0$. This means that the matrix we're looking for needs to have $(0, 1, 0)$ and $(2, 3, -5)$ as rows, because we need:

$$A \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

if and only if $y = 0$ and $2x + 3y - 5z = 0$. For the third row of A, we can put $(0, 0, 0)$, because we don't need to impose any extra requirements on x, y, z. So we conclude that the answer is:

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 2 & 3 & -5 \\ 0 & 0 & 0 \end{bmatrix}$$

Note that other choices of the matrix A would have also worked, e.g. the third row could have been any linear combination of the other two rows.

Grading Rubric:

- Correct matrix with explanation as to why the nullspace is the intersection we need *(10 points)*
- Correct matrix with no explanation *(7 points)*
- Incorrect matrix, but its nullspace contains at least one of the two planes involved *(3 points)*
- Missing or wrong answer *(0 points)*

(2) The problem boils down to finding vectors $\boldsymbol{b}$ which do or do not lie in the column space of A. Since any vector in the column space of A has its third entry 0, then:

$$\boldsymbol{b} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

has the property that $A\boldsymbol{v} = \boldsymbol{b}$ has no solutions. On the other hand, for:

$$\boldsymbol{b} = \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}$$

the system $A\boldsymbol{v} = \boldsymbol{b}$ has a solution (in fact, it suffices to take $\boldsymbol{v} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$). Therefore, it has infinitely many solutions, because we can add to $\boldsymbol{v}$ any multiple of a vector lying in the intersection in part (1), and the equation $A\boldsymbol{v} = \boldsymbol{b}$ will continue to hold.

Grading Rubric (for each of the two bullets):

- Correct vector with explanation *(5 points)*
- Correct vector without explanation *(3 points)*
- Incorrect vector *(0 points)*

For the second bullet of the problem, subtract an extra point if the student only shows that there exists a solution, but doesn't argue that there are infinitely many solutions.

Problem 3:

(1) Compute the reduced row echelon form of the matrix:

$$A = \begin{bmatrix} 0 & -1 & 2 & 1 \\ 1 & 2 & -3 & 0 \\ 1 & 3 & -5 & -1 \end{bmatrix}$$

(all zero rows should be at the bottom of A). *(10 points)*

(2) Use the result of part (1) to find the full set of solutions to the equation:

$$A \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = 0$$

(10 points)

Solution: (1)

$$\begin{bmatrix} 0 & -1 & 2 & 1 \\ 1 & 2 & -3 & 0 \\ 1 & 3 & -5 & -1 \end{bmatrix} \xrightarrow{swap} \begin{bmatrix} 1 & 2 & -3 & 0 \\ 0 & -1 & 2 & 1 \\ 1 & 3 & -5 & -1 \end{bmatrix} \xrightarrow{r_3 - r_1} \begin{bmatrix} 1 & 2 & -3 & 0 \\ 0 & -1 & 2 & 1 \\ 0 & 1 & -2 & -1 \end{bmatrix} \xrightarrow{r_3 + r_2}$$

$$\begin{bmatrix} 1 & 2 & -3 & 0 \\ 0 & -1 & 2 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{r_2(-1)} \begin{bmatrix} 1 & 2 & -3 & 0 \\ 0 & 1 & -2 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{r_1 - 2r_2} \begin{bmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & -2 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Grading Rubric:

- Correct answer with all the steps *(10 points)*
- -2 points for minor computational mistake
- -1 points if the row of zeroes isn't at the bottom
- Significant computational mistakes, but correct process *(5 points)*
- Student did row echelon form, instead of reduced row echelon form, but correctly *(5 points)*
- Incorrect process *(0 points)*

(2) The solutions to a system of equations do not change under row operations, so we may as well solve the system corresponding to the reduced row echelon form:

$$\begin{cases} a + c + 2d = 0 \\ b - 2c - d = 0 \end{cases}$$

The pivot variables are a, b and the free variables are c, d. So the general solution is obtained by letting the free variables be arbitrary, and determining what the pivot variables must be based on the equations above:

$$\begin{bmatrix} -c - 2d \\ 2c + d \\ c \\ d \end{bmatrix}$$

Grading Rubric:

- Correct answer with all the steps *(10 points)*
- Student provides a basis of the nullspace, but doesn't say that the general solution is a linear combination of the basis vectors *(8 points)*
- Student provides a correct description of the nullspace, but without using the reduced row echelon form *(5 points)*
- Incorrect process *(0 points)*

If the student is using the wrong reduced row echelon form due to a computational mistake, please accept their wrong form as input and grade their process.

Problem 4:

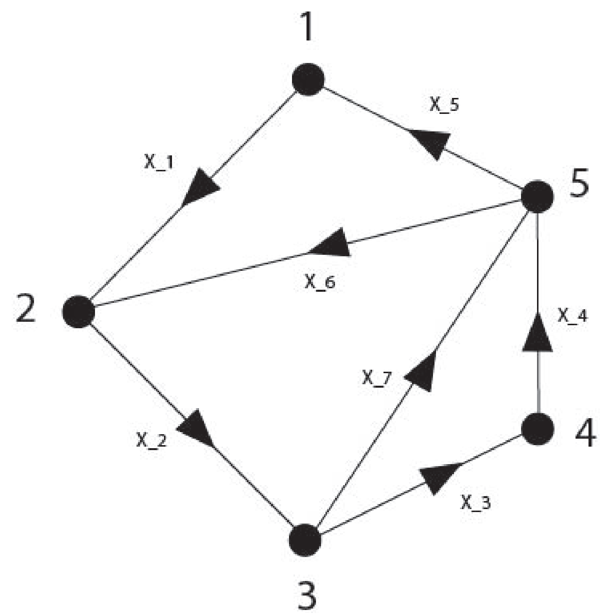
The diagram below represents 5 nodes (represented by the circles) connected by 7 pieces of conducting wire (represented by the lines). The intensity of the electric current flowing through these pieces of wire is $x_1, ..., x_7$, in the direction of the arrow. If any of the x_i's are negative, this just means current flowing in opposite direction to the arrow.

Kirchoff's first law says that, at every node, the incoming current should equal the outgoing current.

(1) Write down explicitly the incidence matrix of the diagram. By definition, this is the 5×7 matrix A whose entry at row i and column j is:

$$\begin{cases} 1 & \text{if the current on the } j\text{–th wire flows into node } i \\ -1 & \text{if the current on the } j\text{–th wire flows out of node } i \\ 0 & \text{if the } j\text{–th wire does not intersect node } i \end{cases}$$

(the j–th wire is the one denoted by the variable x_j in the diagram). *(5 points)*



(2) Express Kirchoff's first law as a linear algebra condition on the vector of currents $\begin{bmatrix} x_1 \\ \vdots \\ x_7 \end{bmatrix}$, which involves the incidence matrix A (justify). *(5 points)*

(3) By using the reduced row echelon form of A, find all possible vectors of currents $\begin{bmatrix} x_1 \\ \vdots \\ x_7 \end{bmatrix}$ which satisfy Kirchoff's law for the diagram above. *(10 points)*

Solution: (1) The incidence matrix is

$$\begin{bmatrix} -1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & -1 & 1 \end{bmatrix}$$

Grading Rubric:

- -1 points for each incorrect entry
- -2 points if the roles of 1 and -1 are swapped

(2) The condition is:

$$A \begin{bmatrix} x_1 \\ \vdots \\ x_7 \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$$

Comparing first rows on both sides of this equation, for example, states that $-x_1+x_5=0$, and this corresponds to the condition that incoming current equal outgoing current in node 1. Similarly, the i–th row of the equation enforces Kirchoff's law at node i.

Grading Rubric:

- Correct condition with justification (along the lines of the previous sentence) *(5 points)*
- Correct condition without justification *(3 points)*
- Incorrect condition *(0 points)*

(3) The row reduced form of the matrix is:

$$\begin{bmatrix} 1 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & -1 & 0 \\ 0 & 0 & 1 & 0 & -1 & -1 & 1 \\ 0 & 0 & 0 & 1 & -1 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

(note that we expect students to show all the steps, but we do not do so here, because we have already showed the process in the previous problem). The pivot variables are x_1, x_2, x_3, x_4 and the free variables are x_5, x_6, x_7, so the general form of possible vectors of currents is:

$$\begin{bmatrix} x_5 \\ x_5 + x_6 \\ x_5 + x_6 - x_7 \\ x_5 + x_6 - x_7 \\ x_5 \\ x_6 \\ x_7 \end{bmatrix}$$

Grading Rubric:

- Correct answer with all the row reduction steps *(10 points)*
- Minor algebra mistakes *(7-8 points)*
- Correct answer without row reduction steps explicitly shown *(5 points)*
- Incorrect answer but correct row reduction with all the steps *(5 points)*
- Missing or significantly incorrect answer *(0 points)*

Problem 5: (justify all your answers)

(1) If X is an invertible square matrix, what can you say about $C(X)$ and $N(X)$? *(10 points)*

(2) If $Y = \left[\begin{array}{c} A \\ \hline B \end{array}\right]$ is a block matrix, what is $N(Y)$ in terms of $N(A)$ and $N(B)$? *(5 points)*

(3) If $Z = \left[\begin{array}{c|c} A & B \end{array}\right]$ is a block matrix, what is $C(Z)$ in terms of $C(A)$ and $C(B)$? *(5 points)*

Solution: (1) Suppose X is $n \times n$. First, we claim that $N(X)$ consists only of the origin, so $N(X) = \{0\}$. To see this, suppose that $X\boldsymbol{v} = 0$. Then $X^{-1}X\boldsymbol{v} = X^{-1}0$, and so $\boldsymbol{v}$ must be 0.

Next, we claim that $C(X)$ must be the entire space $\mathbb{R}^n$. Indeed, if $\boldsymbol{w}$ is any vector in $\mathbb{R}^n$, we can write $X(X^{-1}\boldsymbol{w}) = \boldsymbol{w}$, and so the entries of $X^{-1}\boldsymbol{w}$ tell us what linear combination of columns of X produces $\boldsymbol{w}$.

Grading Rubric (for each of $C(X)$ and $N(X)$):

- Correct answer with explanation *(5 points)*
- Correct answer without explanation *(3 points)*
- Incorrect answer *(0 points)*

(2) $N(Y)$ is the intersection of $N(A)$ and $N(B)$. In other words, a vector $\boldsymbol{v}$ is in $N(Y)$ if and only if both $A\boldsymbol{v} = 0$ and $B\boldsymbol{v} = 0$. To see this, note that the product $Y\boldsymbol{v}$ will just be:

$$Y\boldsymbol{v} = \left[\begin{array}{c} A \\ \hline B \end{array}\right] \boldsymbol{v} = \left[\begin{array}{c} A\boldsymbol{v} \\ \hline B\boldsymbol{v} \end{array}\right]$$

and this is clearly 0 if and only if $Av = Bv = 0$.

Grading Rubric:

- Correct answer with explanation *(5 points)*
- Correct answer without explanation *(3 points)*
- Incorrect answer *(0 points)*

(3) $C(Z)$ is the sum of $C(A)$ and $C(B)$. By this, we mean that a vector $\boldsymbol{v}$ is in $C(Z)$ if and only if $\boldsymbol{v}$ can be written as a sum $\boldsymbol{v} = \boldsymbol{w}_1 + \boldsymbol{w}_2$, where $\boldsymbol{w}_1$ is in the column space of A and $\boldsymbol{w}_2$ is in the column space of B. Indeed, the column space of Z consists of linear combinations of the columns of Z, and such a combination can be broken up into the sum of a linear combination of the columns of A and a linear combination of columns of B.

Grading Rubric:

- Correct answer with explanation *(5 points)*
- Correct answer without explanation *(3 points)*
- Incorrect answer *(0 points)*